# Vector Calculus

Vector field transformation from cylindrical $(\bar{V} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z})$ to cartesian $(\bar{V} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z})$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Vector field transformation from cartesian $(\bar{V} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z})$ to cylindrical $(\bar{V} = A_\rho \hat{\rho} + A_\phi \hat{\phi} + A_z \hat{z})$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- Vector field transformation from spherical to cartesign
$\bar{V} = (A_\rho \hat{\rho} + A_\theta \hat{\theta} + A_\phi \hat{\phi}) \longrightarrow (\bar{V} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z})$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{array}{c|ccc} & \sin\theta & \cos\theta & \frac{d}{d\phi} \\ \hline \cos\phi & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\phi & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \frac{d}{d\theta} & \cos\theta & -\sin\theta & 0 \end{array} \cdot \begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix}$$

Vector field transformation from cartesign to cylindrical
$\bar{V} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \longrightarrow (\bar{V} = A_\rho \hat{\rho} + A_\theta \hat{\theta} + A_\phi \hat{\phi})$

$$\begin{bmatrix} A_\rho \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

A vector field can be maid in terms of $(\rho, \phi, z)$ or $(x, y, z)$ or $(\rho, \theta, \phi)$ without changing the 3 primary field axes.

ie) $\bar{V} = A_x(x,y,z)\,\hat{a_x} + A_y(x,y,z)\,\hat{a_y} + A_z(x,y,z)\,\hat{a_z}$

becomes $\bar{V} = A_x(\rho,\theta,\phi)\,\hat{a_x} + A_y(\rho,\theta,\phi)\,\hat{a_y} + A_z(\rho,\theta,\phi)\,\hat{a_z}$

This is done by ordinary polar or spherical transformation.

| Cordinate System | Cartesign | Cylindrical | Spherical |
|---|---|---|---|
| Base Vectors | $\hat{a}_x \hat{a}_y \hat{a}_z$ | $\hat{a}_\rho \hat{a}_\phi \hat{a}_z$ | $\hat{a}_r \hat{a}_\theta \hat{a}_\phi$ |
| Differential length | | | |
| $dl_1$ | $dx\,\hat{a}_x$ | $d\rho\,\hat{a}_\rho$ | $dr\,\hat{a}_r$ |
| $dl_2$ | $dy\,\hat{a}_y$ | $\rho\,d\phi\,\hat{a}_\phi$ | $r\,d\theta\,\hat{a}_\theta$ |
| $dl_3$ | $dz\,\hat{a}_z$ | $dz\,\hat{a}_z$ | $r\sin\theta\,d\phi\,\hat{a}_\phi$ |
| Metric Coefficient | | | |
| $h_1$ | 1 | 1 | 1 |
| $h_2$ | 1 | $\rho$ | $r$ |
| $h_3$ | 1 | 1 | $r\sin\theta$ |

**Gradient** Scalar input Vector output

$\nabla f$

In general,
$$\boxed{\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\hat{a}_{u_1} + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\hat{a}_{u_2} + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\hat{a}_{u_3}}$$

Cartesign
$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

Cylindrical
$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$$

Spherical
$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

**Gradient identities** ($U, V$ scalar fields)

- $\nabla(U+V) = \nabla U + \nabla V$
- $\nabla(UV) = U(\nabla V) + V(\nabla U)$
- $\nabla\left(\frac{U}{V}\right) = \frac{V\nabla U - U\nabla V}{V^2}$
- $\nabla V^n = nV^{n-1}\nabla V$

**Properties**

→ Magnitude of $\nabla f$ = maximum rate of change in $f$ per unit distance

→ $\nabla f$ direction = direction of max rate of change

→ $\nabla f$ perpendicular to $f = c$ surface eq equipotential surfaces

→ if $\bar{A} = \nabla f$ then $f$ is scalar potential of A

**Directional derivative of V along $\hat{a}$**

$$\boxed{\frac{dV}{dl} = \nabla V \cdot \hat{a}}$$

→ The projection of $\nabla V$ in the direction of a unit vector $\hat{a}$ is called the directional derivative of V along $\hat{a}$ which gives rate of change of V along $\hat{a}$.

→ output of $\nabla V \cdot \hat{a}$ = scalar

**Divergence** → Defined as the net outflow of flux from infinitely small closed surface around the point per unit volume.

(Vector input Scalar o/p)

$$\nabla \cdot A = \lim_{\Delta V \to 0} \frac{\oint_S A \cdot ds}{\Delta V}$$

$$\boxed{\nabla \cdot A = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_{u_1}) + \frac{\partial}{\partial u_2}(h_1 h_3 A_{u_2}) + \frac{\partial}{\partial u_3}(h_1 h_2 A_{u_3})\right]}$$

Cartesian $\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Cylindrical $\nabla \cdot A = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial \phi}(A_\phi) + \frac{\partial}{\partial z}(A_z r)\right] = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

Spherical $\nabla \cdot A = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta A_r) + \frac{\partial}{\partial \theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial \phi}(r A_\phi)\right] = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(A_\phi)}{\partial \phi}$

## Properties

i) $\nabla \cdot (A+B) = \nabla \cdot A + \nabla \cdot B$

ii) $\nabla \cdot V\bar{A} = V\,\nabla \cdot \bar{A} + \bar{A} \cdot \nabla V$

$\oint_S \bar{A} \cdot d\bar{s} = \oint_V \nabla \cdot \bar{A}\, dV$

$\nabla \cdot \nabla A = \nabla^2 A$ $A \rightarrow$ scalar field

Laplacian $= \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$

$\nabla \cdot \vec{A} = 0 \Rightarrow \vec{A}$ Solenoid field

$\rightarrow$ Every solenoidal field can be expressed as the curl of a another vector field. ie) $\nabla \cdot (\nabla \times A) = 0$ Rotation has no source or sink.

## Curl Vector $\rightarrow$ Vector

Curl of any vector field, is a rotational vector whose magnitude is the maximum circulation of that vector per unit area as the area tends to zero and whose direction is normal to the plane containing A, when the area is oriented to make the circulation maximum.

$$\boxed{\text{Curl } A = \nabla \times A = \lim_{\Delta s \to 0} \frac{\oint_L A \cdot dl}{\Delta s}\, \hat{a}_n}$$

In general curl $A = \nabla \times A = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{a}_{u_1} & h_2 \hat{a}_{u_2} & h_3 \hat{a}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_{u_1} & h_2 A_{u_2} & h_3 A_{u_3} \end{vmatrix}$

## Properties

$\nabla \times (\bar{A} + \bar{B}) = \nabla \times \bar{A} + \nabla \times \bar{B}$

$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

$\nabla \times (V\vec{A}) = V(\nabla \times \vec{A}) + \nabla V \times \bar{A}$

$\nabla \times (\nabla V) = 0 \rightarrow$ The gradient of a scalar field has no rotation

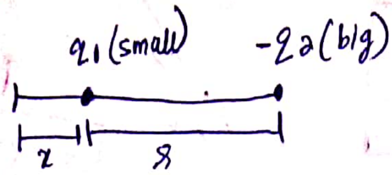
$\bar{A} \times \bar{B} \times \bar{C} = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$

$\nabla \times \nabla \times \bar{A} = \bar{\nabla}(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}$

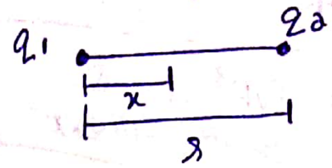
Every ~~gradient~~ irrotation field can be expressed as the divergence of a scalar field.

# Results.

1) Electrostatic Equilibrium



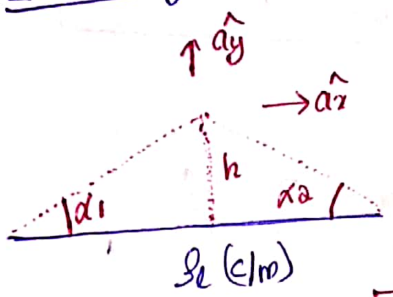

Same polarity




opposite polarity

| | |
|---|---|
| $x = \dfrac{r}{\sqrt{\frac{q_2}{q_1}}+1}$ | $Q = \dfrac{q_1 q_2}{(\sqrt{q_1}+\sqrt{q_2})^2}$ |
| $x = \dfrac{r}{\sqrt{\frac{q_2}{q_1}}-1}$ | $Q = \dfrac{q_1 q_2}{(\sqrt{q_1}-\sqrt{q_2})^2}$ |

2) Line charge distribution




$$\boxed{E_x = \frac{\rho_L}{4\pi\varepsilon_0 h}\left[\sin\alpha_2 - \sin\alpha_1\right]\hat{a_x} \qquad E_y = \frac{\rho_L}{4\pi\varepsilon_0 h}\left[\cos\alpha_1 + \cos\alpha_2\right]\hat{a_y}}$$

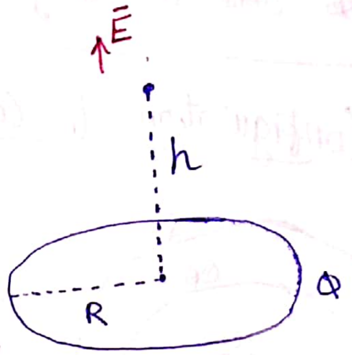
3) Infinite long line

$$\boxed{E_y = \frac{\rho_L}{2\pi\varepsilon_0 h}\hat{a_y}}$$

4) Circular Ring

$$\boxed{E = \frac{Qh}{4\pi\varepsilon_0 (R^2+h^2)^{3/2}}}$$

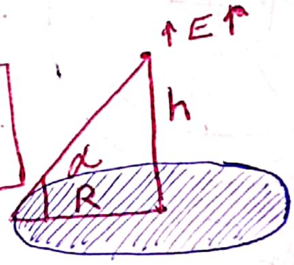
$$\boxed{E_{max} \Rightarrow h = \pm R/\sqrt{2}}$$




$$\boxed{P = \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+h^2}}}$$

5) Circular disc

$$\boxed{E = \frac{\rho_s}{2\varepsilon_0}(1-\sin\alpha)}$$




6) Infinite plane $\boxed{E = \dfrac{\rho_s}{2\varepsilon_0}}$

7) Hollow Non conducting sphere / hollow or solid conductor

$0 < x < R$

$$\boxed{E = 0 \qquad V = \frac{Q}{4\pi\varepsilon R}}$$

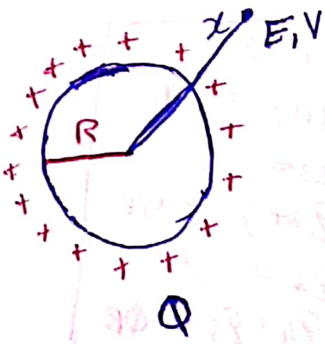
$x > R$

$$\boxed{E = \frac{Q}{4\pi\varepsilon x^2}\hat{a_r} \qquad V = \frac{Q}{4\pi\varepsilon x}}$$

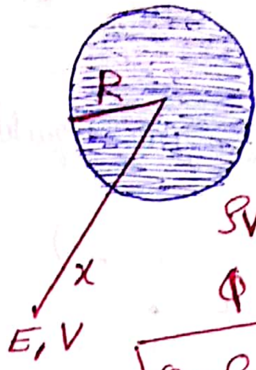
Total potential energy of configuration or work done to achieve configuration

$$\boxed{U = \frac{1}{4\pi\varepsilon_0 R}\cdot\frac{Q^2}{2}}$$

$$\boxed{U = \int_0^Q V\,dq}$$



## 8) Solid Non-Conducting Sphere




$\rho_v\ C/m^3$

$Q$

$\boxed{Q = \rho_v \frac{4}{3}\pi R^3}$

**$0 < x < R$**

$$\boxed{E = \frac{x\rho_v}{3\varepsilon} = \frac{Qx}{4\pi\varepsilon R^3}}$$

$$V = \frac{\rho_v}{3\varepsilon}\left[\frac{3R^2 - x^2}{2}\right]$$

$$V = \frac{Q}{4\pi\varepsilon R^3}\left[\frac{3R^2 - x^2}{2}\right]$$

**$x > R$**

$$\boxed{E = \frac{\rho_v R^3}{3\varepsilon x^2} = \frac{Q}{4\pi\varepsilon_0 x^2}}$$

$$\boxed{V = \frac{\rho_v R^3}{3\varepsilon x} = \frac{Q}{4\pi\varepsilon x}}$$

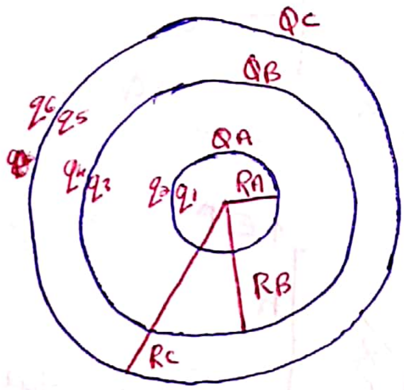
$$V_{surface} = \frac{\rho_v R^2}{3\varepsilon} = \frac{Q}{4\pi\varepsilon R}$$

$$V_{centre} = \frac{\rho_v R^2}{2\varepsilon} = \frac{3 \cdot Q}{2 \cdot 4\pi\varepsilon R}$$

$\Rightarrow \boxed{V_{centre} = 1.5\, V_{surface}}$

Work done in energising the volume configuration (potential energy) $= \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 R}$

## 9) Configuration of Concentric conducting spheres



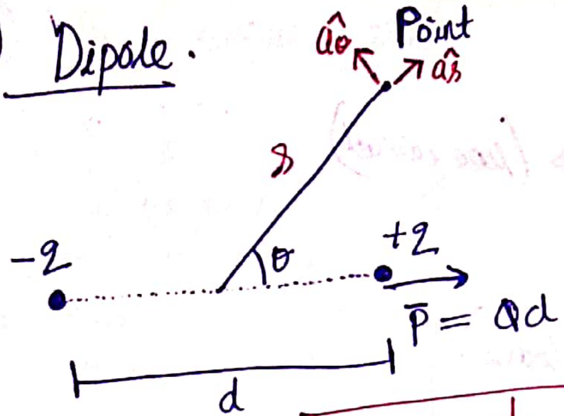

| | |
|---|---|
| $x > R_C$ | $E = \frac{Q_A}{4\pi\varepsilon_0 x^2} + \frac{Q_B}{4\pi\varepsilon_0 x^2} + \frac{Q_C}{4\pi\varepsilon_0 x^2}$ |
| $R_B < x < R_C$ | $E = \frac{Q_B}{4\pi\varepsilon_0 x^2} + \frac{Q_A}{4\pi\varepsilon_0 x^2}$ |
| $R_A < x < R_B$ | $E = \frac{Q_A}{4\pi\varepsilon_0 x^2}$ |
| $x > R_C$ | $V = \frac{Q_A}{4\pi\varepsilon_0 x} + \frac{Q_B}{4\pi\varepsilon_0 x} + \frac{Q_C}{4\pi\varepsilon_0 x}$ |
| $R_B < x < R_C$ | $V = \frac{Q_A}{4\pi\varepsilon_0 x} + \frac{Q_B}{4\pi\varepsilon_0 x} + \frac{Q_C}{4\pi\varepsilon_0 R_C}$ |
| $R_A < x < R_B$ | $V = \frac{Q_A}{4\pi\varepsilon_0 x} + \frac{Q_B}{4\pi\varepsilon_0 R_B} + \frac{Q_C}{4\pi\varepsilon_0 R_C}$ |
| $0 < x < R_A$ | $V = \frac{Q_A}{4\pi\varepsilon_0 R_A} + \frac{Q_B}{4\pi\varepsilon_0 R_B} + \frac{Q_C}{4\pi\varepsilon_0 R_C}$ |

$$V_A - V_B = \frac{Q_A}{4\pi\varepsilon_0}\left[\frac{1}{R_A} - \frac{1}{R_B}\right]$$

$q_1 = 0$
$q_1 + q_2 = Q_A$
$q_3 = -q_2$
$q_3 + q_4 = Q_B$
$q_5 = -q_4$
$q_4 + q_6 = Q_B$

if there is charge enclosed within the innermost sphere then
$q_1 = -q_{enclosed}$

## 10) Dipole.




$$\boxed{E = \frac{|p|}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta\right] \qquad |E| = \frac{|p|}{4\pi\varepsilon_0 r^3}\sqrt{1+3\cos^2\theta}}$$

$$\boxed{V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

$\boxed{\text{If } r >> d.}$

**Along dipole axis**

End Fire direction

$$\boxed{|E| = \frac{2|p|}{4\pi\varepsilon_0 r^3}} \qquad \boxed{V = \frac{p}{4\pi\varepsilon_0 r^2}}$$

**Perpendicular to dipole axis**

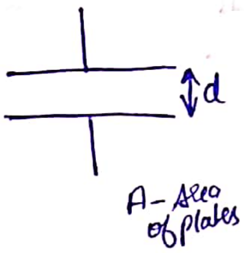
Broad Fire direction.

$$\boxed{|E| = \frac{|p|}{4\pi\varepsilon_0 r^3}} \qquad \boxed{V = 0}$$

## Capacitance

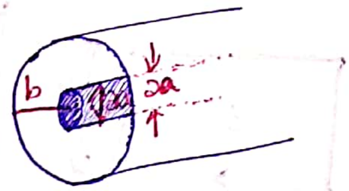
### ① Parallel plate capacitor

~~Capacitor~~



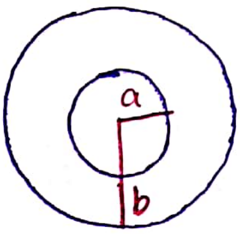

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

### ② Coaxial cable




$$\boxed{C = \frac{2\pi\varepsilon l}{\ln\left(\frac{b}{a}\right)}}$$

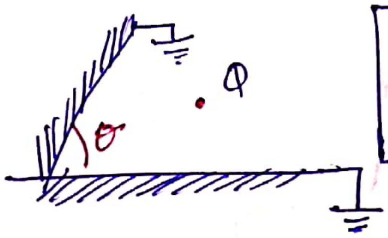
### ③ Spherical capacitor




$$\boxed{C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$$

→ Capacitors can be splitted in parallel or in series.
→ Parallel capacitors $\int E \cdot dl$ equal
→ series capacitor charge equal
→ No matter the splitting the $\bar{D}$ inside is always constant
$\bar{D} = (\rho_s$ of any one plate$)$

## Concept of images




$$\boxed{\text{No. of images} = \frac{360}{\theta} - 1}$$

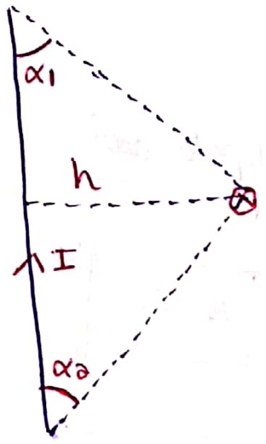
## ~~Elec~~ Energy density

$$E_d = \frac{1}{2}\bar{D}\cdot\bar{E} = \frac{1}{2}\varepsilon E\cdot E$$

$$E_d = \frac{1}{2}\varepsilon E^2$$

# Magnetic field results

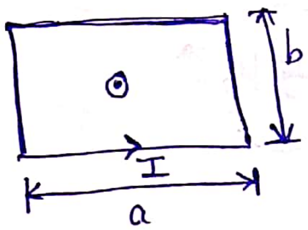
1) Field due to straight line conductor (line current)




$$\boxed{H = \frac{I}{4\pi h}\left(\cos\alpha_1 + \cos\alpha_2\right)}$$

For infinitely long $\longrightarrow$ $\boxed{H = \frac{I}{2\pi h}}$

2) Rectangular loop




$$\boxed{H = \frac{2I}{\pi}\frac{\sqrt{a^2+b^2}}{ab}}$$

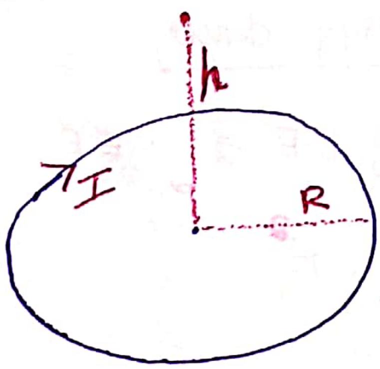
H if square of side a

$$\boxed{H = \frac{2\sqrt{2}I}{\pi a}}$$

3) For a n sided regular polygon of side d.

$$\boxed{H = \frac{nI}{\pi d}\tan\left(\frac{180}{n}\right)\sin\left(\frac{180}{n}\right)}$$

4) Circular loop




$$\boxed{H = \frac{IR^2}{2\left(R^2+h^2\right)^{3/2}}}$$

$$\boxed{H_{centre} = \frac{I}{2R}}$$

$$\boxed{H_{centre\ arc\ of\ \theta} = \frac{I}{2R}\left(\frac{\theta}{2\pi}\right)}$$

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## 5) Infinite surface current

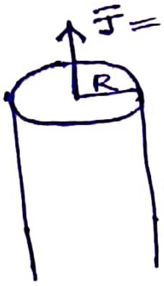
$$\boxed{H = \frac{K}{2}}$$

$$\bar{K} = \frac{I\,dl}{Area}$$

$$K = \frac{I\,d\bar{l}}{dl \times l} = \frac{I}{l}\ (A/m)$$

## 6) Volume current

$$\bar{J} = \frac{I\,dl}{Volume} = \frac{I\,dl}{dl \times Area} = \frac{I}{Area}\ (A/m^2)$$



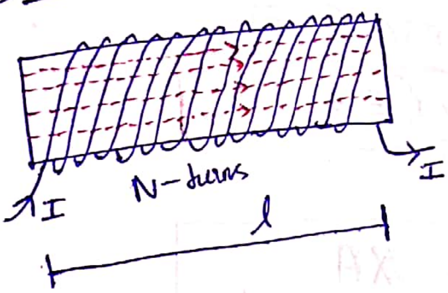

$$\boxed{H_{inside} = \frac{I x}{2\pi R^2}} \quad (0 < x < R)$$

$$\boxed{I = J \times Area}$$

$$\boxed{I = J \pi R^2}$$

$$\boxed{H_{outside} = \frac{I}{2\pi x}} \quad x > R$$

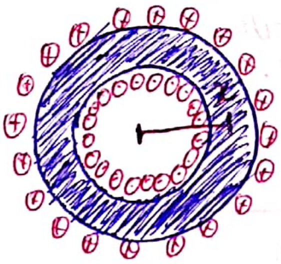
## 7) Solenoid:




$$\boxed{H = \frac{NI}{l}} \qquad \boxed{L = \frac{\mu N^2 A}{l}}$$

Field outside = 0

## 8) Toroid



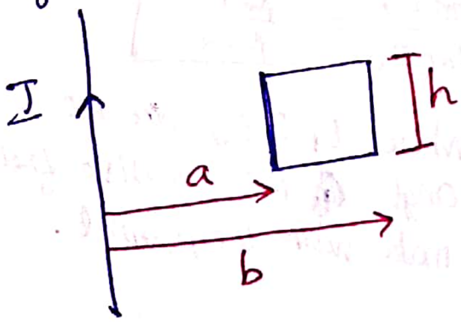

$x \rightarrow$ distance from centre

inner radius $r$ $\quad r < x < R$

outer radius $R$

$$\boxed{H = \frac{NI}{2\pi x}}$$

## 9) flux due to line current




$$\boxed{\phi = \frac{\mu I h}{2\pi} \ln \frac{b}{a}}$$

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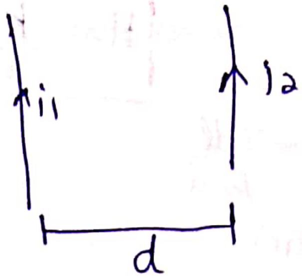
10) Force between current carrying conductors $I_1, I_2$

Force per unit length.

$$\boxed{F/_{l} = \frac{\mu i_1 i_2}{2\pi d}}$$




Current same dir. Attraction

Opposite dir. Repulsion.

## Maxwell's equations for Time varying fields

$$\boxed{\begin{aligned} \nabla\cdot D &= \rho_V \\ \nabla\times E &= -\frac{\partial B}{\partial t} \\ \nabla\cdot B &= 0 \\ \nabla\times H &= J_D + \frac{\partial D}{\partial t} \end{aligned}}$$

Poissons Equation.

$$\boxed{\nabla^2 V = \frac{-\rho_V}{\varepsilon}}$$

Equation of Continuity.

$$\boxed{\nabla\cdot J = -\frac{\partial \rho_V}{\partial t}}$$

Lorentz's Force Equation

$$\boxed{F = q(E + \vec{v}\times B)}$$

Electric potential

$$\boxed{E = -\nabla V - \frac{\partial A}{\partial t}}$$

$$\boxed{\begin{aligned} B &= \nabla\times A \\ \nabla\cdot A &= -\mu\varepsilon\frac{\partial V}{\partial t} \end{aligned}}$$

Magnetic Vector potential.

## Boundary Conditions

$E_{1t} = E_{2t}$

$D_{2n} - D_{1n} = \rho_s$

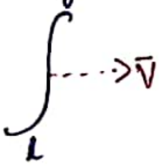
$H_{2t} - H_{1t} = K_s$

$B_{2n} = B_{1n}$

$$\boxed{\begin{aligned} \frac{\tan\theta_1}{\tan\theta_2} &= \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \\ \frac{\tan\theta_1}{\tan\theta_2} &= \frac{\mu_{r1}}{\mu_{r2}} \end{aligned}}$$

Where $\theta_1$ & $\theta_2$ are the angles the respective fields make with the normal.

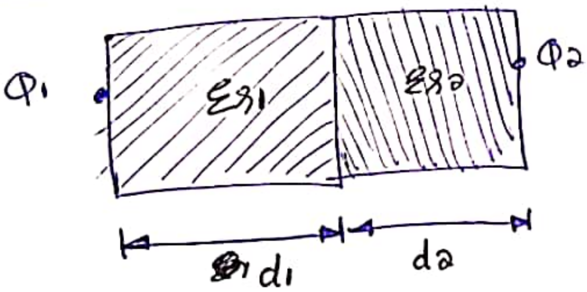
Emf indued in a wire moving in a magnetic field $\bar{B}$ with velocity $\bar{V}$




$$e = \int_l (\bar{V} \times \bar{B}) \cdot dl$$

Force experienced by a current carrying conductor on a magnetic field

$$F = \int I (dl \times B)$$




$$\boxed{F_{12} = \frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2}{\left(d_1\sqrt{\varepsilon_{r1}} + d_2\sqrt{\varepsilon_{r2}}\right)^2}}$$

magnetic moment of a circular loop (1 turn)

$$m = IA$$

$A$ = area
$I$ = current in loop

## ★ Important problems

If $V = 2$ Volts at $x = 1mm$ and $V = 0$ @ $x = 0$ and $\rho_v = -10^{6}\varepsilon_0 \, C/m^3$, then find the magnitude of electric field intensity in free space at $x = 1mm$.

$$-\nabla^2 V = \frac{\rho_v}{\varepsilon_0} \Rightarrow \frac{\partial^2 V}{\partial x^2} = 10^6$$

$$\frac{\partial V}{\partial x} = 10^6 x + c_1$$

$$V = \frac{10^6 x^2}{2} + c_1 x + c_2$$

Applying boundary condition

$$V(0) = 0 \Rightarrow c_2 = 0$$

$$V(1 \times 10^{-3}) \Rightarrow 2 \Rightarrow 2 = 0.5 + c_1 10^{-3} = 2$$

$$c_1 = \frac{1.5}{10^{-3}} = 1500$$

$$\therefore V(x) = 5 \times 10^5 x^2 + 1500x$$

$$E(x) = -\nabla V = -\frac{\partial V(x)}{\partial x}\hat{a}_x = -\left(10^6 x + 1500\right)$$

$$E(x = 10^{-3}) = -(1000 + 1500) = \underline{-2500 \, kV/mm}.$$

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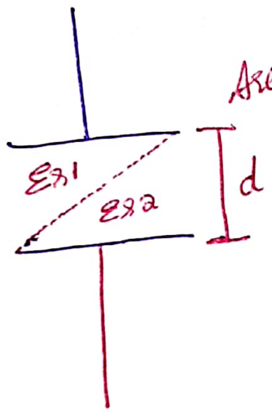
*




Area of plate A

$$\boxed{C_{eff} = \frac{\varepsilon_0 A}{d} \frac{\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r2} - \varepsilon_{r1}} \ln\left(\frac{\varepsilon_{r2}}{\varepsilon_{r1}}\right)}$$

---

$B = \nabla \times \bar{A}$     $A \rightarrow$ magnetic vector potential

$\nabla \cdot \bar{A} = 0$ for static fields.

$$\nabla^2 \bar{A} = -\nabla \times \nabla \times A + \nabla(\nabla \cdot A)$$

$$\boxed{\nabla^2 A = -\mu J}$$

## Multiple integrals conclusions

→ Identify the required or given order of integration by looking at the limit

→ Multiple integral is evaluated first w.r.t the variable whose limits are a function of the remaining variables, that has not been integrated yet and last w.r.t the variable with constant limits.

→ If the given function is not directly integrable in the given order, it might be integrable in a different order in some special cases.

→ In this case try to plot the region and change the order of integration.

→ To identify whether change of variables is needed, observe the function and the limit values.

---

## Parametric Equation

→ In mathematics, a parametric equation defines a group of quantities as functions of one or more independent variables called parameters.

→ They are commonly used to express coordinates of points that make up a geometric object (eg curve, surface) in which case the equations are collectively called a parametric representation or parameterization

eg:- $\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\}$ parametric representation of unit circle.

*Note: Parametric equations are commonly used in kinematics where the trajectory of an object is represented by equations depending on time as parameter.

→ Parameterizations are non unique; More than one set of parametric equation can specify the same curve.

### Representation of Planar Curves

| Type | Form | Example | Description |
|---|---|---|---|
| 1) Explicit | $y = f(x)$ | $y = mx + b$ | Line |
| 2) Implicit | $f(x,y) = 0$ | $(x-a)^2 + (y-b)^2 = r^2$ | Circle |
| 3) Parametric | $x = \frac{g_1(t)}{g_2(t)}$ <br> $y = \frac{h(t)}{h_2(t)}$ | $x = a_0 + a_1 t$ <br> $y = b_0 + b_1 t$ | Line |
| | | $x = a + r\cos t$ <br> $y = b + r\sin t$ | Circle. |

## Implicitization

→ Converting a set of parametric equation to a single implicit equation involves eliminating the variable $t$ from the simultaneous equations.

→ If any of the equations can be solved for $t$, then the expression obtained can be substituted an equation involving other quantities only.

## Parametric representation of curves in 3D

$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

Where $\bar{r}(t)$ is the position vector of any point on the curve.

A point is on the curve if it can satisfy this equation for atleast one value of $t$

i.e if $(a,b,c)$ is on curve

then $a = x(t_0)$, $b = y(t_0)$, $c = z(t_0)$ — Atleast one $t_0$ simultaneously satisfies this.

## Differentiability.

if $\lim_{\delta t \to 0} \frac{\bar{F}(t+\delta t) - \bar{F}(t)}{\delta t}$ exists then differentiable at $t$.

if $F(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is a parametric representation of curve $C$.

then $\boxed{\frac{dF}{dt} = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}}$ represents the tangent vector to curve $C$.

## Level surface. [Similar to equipotential points]

if $\phi(x,y,z)$ is a scalar point function/Scalar field then all points $(x,y,z)$ satisfying equation $\phi(x,y,z) = c$ (some constant) is called a level surface of $\phi$ at level $c$.

* Note: For different values of $c$ we get different level surfaces and the set of all level surfaces is known as family of level surfaces.

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## Basic Vector calculus Results

I) A line containing the point $(x_0, y_0, z_0)$ and parallel to the vector $\bar{v} = a\hat{i} + b\hat{j} + c\hat{k}$ has parametric equation

Equation of line.

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

if $a, b, c$ are non zero then Symmetric equation of the line can be written as $\Rightarrow$ $\boxed{\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c} = t}$

Standard form

II) A vector $\vec{N}$ If it is normal to every vector in a plane is called the normal vector to the plane.

Equation of plane.

- Equation of a plane containing the point $(x_0, y_0, z_0)$ with normal vector $\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$ is

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Standard form $\boxed{Ax + By + Cz = D}$ — *Any plane can be expressed as this form

III Angle between 2 planes is the angle between their unit vectors.

Angle b/w Vectors.

if $\bar{A}$ & $\bar{B}$ are two vectors then the angle between the vectors is given by $\boxed{\theta = \cos^{-1}\left(\dfrac{\bar{A} \cdot \bar{B}}{|A||B|}\right)}$

IV

Vector Normal to given Vectors

→ if $\bar{A}$ & $\bar{B}$ are two vectors then $\boxed{\dfrac{\bar{A} \times \bar{B}}{|A||B|}}$ is a unit vector normal to both $\bar{A}$ & $\bar{B}$

⇒ if $\bar{N}_1$ & $\bar{N}_2$ are normal vectors to plane$_1$ & plane 2 respectively then the direction vector of intersection of plane must lie in both planes and therefore must be normal to $\bar{N}_1$ & $\bar{N}_2$ simultaneously.

∴ The directional vector giving intersection of two planes with normal $\bar{N}_1$ & $\bar{N}_2$ is $\bar{N}_1 \times \bar{N}_2$.

# Gradient, Curl, Divergence

## Definition

For all definition, properties & concepts for these operators refer cmft book.

☆☆☆

if $Ax+By+Cz=D$ represents a plane

then put $\phi(x,y,z)= Ax+By+Cz$

then $\phi(x,y,z)=c$ represents surface

unit normal to plane $= \boxed{\frac{\nabla\phi}{|\nabla\phi|} = \frac{A\hat{i}+B\hat{j}+C\hat{k}}{\sqrt{A^2+B^2+C^2}}}$

## Properties

→ if $\phi(x,y,z)$ is a scalar function then $\nabla\phi$ is in the direction of greatest rate of increase of $\phi$ and the magnitude is the rate of change (increase of $\phi$) in that direction

→ $\nabla\phi$ at a point $P(x_0,y_0,z_0)$ is normal to the surface given by $\phi(x,y,z)=c$

→ This is because $\phi(x,y,z)=c$ represents the surface or set of all points where $\phi$ is same

∴ There will be no rate of change of $\phi$ in the tangential direction to this surface.

∴ no component of $\nabla\phi$ is along tangent to $\phi(x,y,z)=c$

∴ Since $\nabla\phi$ is along greatest rate of change is achieved normal to this surface.

→ Directional derivative of $\phi$ in the direction of given vector $\bar{a}$ $= \boxed{\nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} \text{ or } \nabla\phi\cdot\hat{a}}$

Very important unit vector

→ Max rate of change of $\phi(x,y,z)$ at $P(x_0,y_0,z_0)$ $\boxed{= |\nabla\phi_{@}|_P}$

→ If $\phi_1(x,y,z)=c_1$ & $\phi_2(x,y,z)=c_2$ are two surfaces then at their point of intersection $P(x_0,y_0,z_0)$

Let $\theta$ be the angle btw two surfaces

$$\boxed{\theta = \cos^{-1}\left[\frac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}\right]}$$

• Equation of tangent plane to the surface $\phi(x,y,z)=c$ @ a point $P(x_0,y_0,z_0)$

with $(\nabla\phi)_{@P} = a\hat{i}+b\hat{j}+c\hat{k}$

then $\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$

* Since $\nabla\phi$ at $P$ is normal to tangent plane and $P$ is a point in plane directly result can be used.

• Equation of normal line to the surface $\phi(x,y,z)=c$ @ point $(P) = P(x_0,y_0,z_0)$

with $\nabla\phi_{@P} = a\hat{i}+b\hat{j}+c\hat{k}$

is $= \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$

* Since $\nabla\phi$ at $P$ is a vector parallel to required line & $P$ is a point on line the direct result can be applied.

## Miscallaneous ★★★★★

- $\boxed{\text{If } \bar{F}(t) \text{ is a vector with constant Magnitude then } \bar{F}\cdot\frac{dF}{dt}=0}$.

proof

Let $\bar{F}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

since $|\bar{F}(t)| = \text{constant}$ $\quad x^2(t) + y^2(t) + z^2(t) = c$

$$2x(t)\frac{dx(t)}{dt} + 2y(t)\cdot\frac{dy(t)}{d(t)} + 2z(t)\cdot\frac{dz(t)}{dt} = 0$$

$$\Rightarrow \boxed{x(t)\cdot\dot{x}(t) + y(t)\,\dot{y}(t) + z(t)\cdot\dot{z}(t) = 0}$$

now. $\dot{\bar{F}}(t) = \frac{d}{dt}F(t) = \dot{x}(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + \dot{z}(t)\,\hat{k}$

$\therefore \bar{F}\cdot\frac{d\bar{F}}{dt} = x\dot{x} + y\dot{y} + z\dot{z} = 0$

---

If $\bar{F}(t)$ is a vector with constant direction then $\bar{F}\times\frac{d\bar{F}}{dt} = 0$

- If $\bar{F}(t)$ is a vector $\frac{d}{dt}F(t)$ has no component ~~port~~ perpendicular to $F(t)$

* if $\bar{F}(t)$ has constant direction $\Rightarrow$ $\bar{F}(t)\times\frac{d}{dt}F(t) = 0$

$\Rightarrow \bar{F}(t)$ & $\frac{d}{dt}F(t)$ is codirectional

* Similarly if $\bar{F}(t)$ has constant magnitude $\frac{d}{dt}F(t)$ has no component parallel to $F(t)$ $\Rightarrow \bar{F}(t)\cdot\frac{d}{dt}F(t) = 0$

$\Rightarrow \bar{F}(t)$ & $\frac{d}{dt}F(t)$ are perpendicular

---

Misc points

- The divergence of the velocity field of a incompressible fluid is zero.
- The velocity potential of a ideal fluid $\phi$ is harmonic $\Rightarrow \nabla^2\phi = 0$

proof Let vector potential $= \phi$

then $\nabla\phi = $ velocity field

but $\nabla\cdot(\nabla\phi) = 0$ [incompressible]

$\Rightarrow \nabla^2\phi = 0$

---

- Harmonic field — Laplacian is zero.

Vector integration

$\bar{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\bar{ds} = \bar{n}\,ds = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$dv = dxdydz$

Line integral can be splitted to separate integrals if path is simple.

Complex integration $\boxed{dz = dx + i\,dy \quad i = \sqrt{-1}}$

## Line integral

- Let $\bar{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a differentiable vector function.
- Then the integral of tangential component of $\bar{A}$ along c form $P_1$ to $P_2$ given by

$$\boxed{\oint_{P_1}^{P_2} \bar{A}\cdot d\bar{l} = \int_{P_1}^{P_2} A_1 dx + A_2 dy + A_3 dz} \qquad \boxed{\bar{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}}$$

Circulation — closed path — counterclockwise +ve.

- If $\bar{A}$ is a force vector acting on a particle which moves from $P_1$ to $P_2$ along curve C then $\oint_{P_1}^{P_2} \bar{A}\cdot d\bar{l}$ gives total work done by Force $\bar{A}$.
- To get work done by particle against force A to move from $P_1$ to $P_2$ $= -\int_{P_1}^{P_2} \bar{A}\cdot dl$

If $\bar{A}$ is conservative or irrotational then $(\nabla\times A = 0)$

$\Rightarrow$ line integral $\int_{P_1}^{P_2} \bar{A}\cdot dl$ is independent of path C.

$$\int_{P_1}^{P_2} \bar{A}\cdot ds = \int_{P_1}^{P_2} \nabla\phi\cdot ds = \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1)$$

where $A = \nabla\phi$

* since every irrotational field can be expressed as gradient of scalar field. $(\nabla\times\nabla\phi) = 0$ gradient is irrotational.

$$\boxed{\text{Ans. if } \nabla\times A = 0 \quad \oint \bar{A}\cdot dl = 0}$$

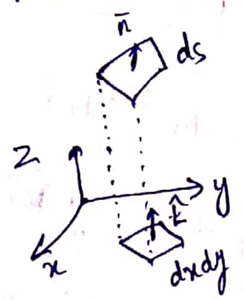
## Surface integral

- $\iint_S (\bar{F}\cdot\bar{n})\,ds = \iint_{R_1} (\bar{F}\cdot\bar{n}) \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$
- $\iint_S (\bar{F}\cdot\bar{n})\,ds = \iint_{R_2} (\bar{F}\cdot\bar{n}) \dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$
- $\iint_S (\bar{F}\cdot\bar{n})\,ds = \iint_{R_3} (\bar{F}\cdot\bar{n}) \dfrac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$

* $\hat{n}$ is the unit vector normal to surface ds

$|\hat{n}\cdot\hat{k}| = |\hat{n}||\hat{k}|\cos\theta = \cos\theta$

$R_1$ projection of S on xy plane
$R_2$ projection of S on yz plane
$R_3$ projection of S on xz plane.




$ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$ds = \dfrac{dx\,dy}{\cos\theta}$

$\theta$ $\angle$ between $\bar{n}$ & $\bar{k}$

## Method to evaluate vector volume & surface integrals

Surface integral

$$\iint_S (\bar{F}\cdot\hat{n})\,ds = \iint_{R_1} \frac{\bar{F}\cdot\hat{n}\;\cancel{dydz}\,dxdy}{|\hat{n}\cdot\hat{k}|}$$

Step I → Find unit vector normal to surface $\hat{n}$

Step II → Find & evaluate $(\bar{F}\cdot\hat{n})$

Step III Find projection of surface S on ~~only~~ any ~~the~~ cordinate plane

Step IV Find limits of plane variables ~~by~~ & from the plane equation.

Step V Find expression of Non plane variable.

Step VI Now surface integration is converted to double integration.

See example 85 & 86.

### Taking limit from plane equation

eg) $2x+3y+6z=12$ → projection of the plane surface thant exist in first quadrant to $xy$ plane.

$x>0,\ y>0\ z>0$ $\quad z = 0 \text{ to } \left(\frac{12-2x-3y}{6}\right)$

$x, y$ limit got by putting $z=0$

$2x+3y=12$ $\quad y = 0 \text{ to } \frac{12-2x}{3}$

$x$ limit got by putting $y=0$

$2x=12$

$x=6, 0$

eg) $\int_V dV$ V is closed

, region bounded by $x=0, y=0, z=0$

$2x+2y+z=4$ — plane.

$z = 4-2x-2y$ //

$y = \frac{4-2x}{2}$ // (at $z=0$)

$x = \frac{4-\cancel{2x}}{2} = 2$ //

### Green's theorm

— If R is a closed Region of the $xy$ plane bounded by C [Closed curve] if $M[x,y]$ & $N[x,y]$ $\frac{\partial N}{\partial x}, \frac{\partial M}{\partial y}$ are continious functions of $x, y$ in R. then

$$\boxed{\oint_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy}$$

Proof

Consider a vector $A = M\hat{i}+N\hat{j}$

then $\oint_C \bar{A}\cdot dl = \oint_C (M\hat{i}+N\hat{j})\cdot(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}) = \oint_C Mdx+Ndy$

But According to stokes $\oint_C \bar{A}\cdot dl = \iint_S \nabla\times A\cdot ds$ $\quad \nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix}$

$\nabla\times A = \hat{k}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]$

$$\oint_C \bar{A}\cdot dl = \int_S (\nabla\times\bar{A})\cdot ds \quad \int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}\left[dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}\right]$$

$$\oint_C A\cdot dl = \boxed{\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy}$$

**Gauss Divergence Theorm**

$$\oiint_S \bar{A}\cdot\bar{n}\,ds = \iiint_V (\nabla\cdot\bar{A})\,dV$$

**Stokes theorm**

$$\oint_C \bar{A}\cdot dl = \iint_S (\nabla\times\bar{A})\cdot\bar{ds}$$

$r$ is the position vector of a point

$\Rightarrow r = x\hat{i} + y\hat{j} + z\hat{k}$

eg) $\oiint_S \bar{r}\cdot ds =$

a) $\frac{1}{2}V$

b) $V$

c) $2V$

d) $3V$

$S \to$ closed surface
$V \to$ closed volume by $S$
$\bar{r} \to$ position vector

Solution $\oiint_S \bar{r}\cdot ds = \iiint_V \nabla\cdot\bar{r}\,dV$ [Gauss divergence theorm]

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$\nabla\cdot r = 1+1+1 = 3$

$\Rightarrow 3\iiint_V dV = \underline{3V}$ option d

# Fourier Series

Standard form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega x) + b_n\sin(n\omega x)$$

period $= T$
$\omega = \frac{2\pi}{T}$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega t)\,dt$$

Even $\to$ only cos + DC
Odd $\to$ only Sine + No DC
Half Wave $\to$ No odd harmonics

$a_0 =$ Average over period

$$\boxed{a_0 = \frac{\text{Area}}{T}}$$

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